3/23/26

Note - we're using the notation $[\vec{v}_1 | \cdots | \vec{v}_n]$ for the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_n$. The book actually uses $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$. We'll stick to ours.

§3.4 <u>Coordinates</u>

You're used to expressing vectors in terms of coordinates —

e.g. $$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = 2\vec{e}_1 + 3\vec{e}_2 + 4\vec{e}_3$$

Note $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ form a <u>basis</u> for $\mathbb{R}^3$.

Try to go beyond this using different bases.

<u>Ex</u> (from book)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$$

Let V be the plane spanned by $\vec{v}_1, \vec{v}_2$. (They form a basis for V.)

Let $\vec{v} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$.

<u>Questions</u>

$$\left.\begin{array}{l} \text{1. Does } \vec{v} \text{ lie in the plane } V\,? \\ \text{2. Is } \vec{v} \text{ a lin comb of } \vec{v}_1, \vec{v}_2\,? \end{array}\right\} \text{ same question}$$

You know how to solve this + we'll skip details.

Answer: Yes!

$$\vec{v} = 3\vec{v}_1 + 2\vec{v}_2$$

New idea: Build up a coordinate system for V based on $\vec{v}_1, \vec{v}_2$.

Note

1. we could do the same with a different basis for V, say $\vec{w}_1, \vec{w}_2$. Then the same $\vec{v}$ would get some other ordered pair instead of $(3,2)$, in terms of $\vec{w}_1, \vec{w}_2$.
   So our coordinate system will depend on the choice of the basis!
   It won't be intrinsic to V.
2. There's no assumption or need for the "axes" (multiples of $\vec{v}_1, \vec{v}_2$ in this case) to be perpendicular.
3. Per book, it's customary to denote bases with capital letters in the Fraktur typeface (see, just above Def 3.4.1). We'll usually use $\mathcal{B}$.
4. Notation (via last example):
   if $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ and $\vec{v} \in V$ then $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
   we say the $\underline{\mathcal{B}\text{-coordinates}}$ of $\vec{v}$ are 3, 2 and the $\underline{\mathcal{B}\text{-coordinate}}$ $\underline{\text{vector}}$ of $\vec{v}$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = [\vec{v}]_{\mathcal{B}}$.
5. if $\mathcal{B}$ is given, then coordinates are uniquely determined. In particular,
   if $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\vec{v} \in V$, and $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$
   then $\vec{v} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$.
6. let $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$. Then $[\vec{v}]_{\mathcal{E}} = \vec{v}$ (abuse of notation a bit)
   e.g. $n=3$, $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, $\vec{v} = \begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix} = 10\vec{e}_1 + 9\vec{e}_2 + 8\vec{e}_3$
   So $[v]_{\mathcal{E}} = \begin{bmatrix} 10 \\ 9 \\ 8 \end{bmatrix} = \vec{v}$

Th 3.4.2 We'll use this shortly

Let $\mathcal{B}$ be a basis for a v.s. V (in $\mathbb{R}^n$).

Let $\vec{x}, \vec{y} \in V$. Then we have linearity:

(a) $$\left[\vec{x} + \vec{y}\right]_{\mathcal{B}} = \left[\vec{x}\right]_{\mathcal{B}} + \left[\vec{y}\right]_{\mathcal{B}}$$

(b) $$\left[k\vec{x}\right]_{\mathcal{B}} = k\left[\vec{x}\right]_{\mathcal{B}} \qquad \forall k \in \mathbb{R} \text{ and } \forall \vec{x} \in V.$$

prove (a) — see book for (b).

Say $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$

$$\vec{x} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n \longleftrightarrow \left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\vec{y} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n \longleftrightarrow \left[\vec{y}\right]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{x} + \vec{y} = (a_1+b_1)\vec{v}_1 + \cdots + (a_n+b_n)\vec{v}_n \longleftrightarrow \left[\vec{x} + \vec{y}\right]_{\mathcal{B}} = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \left[\vec{x}\right]_{\mathcal{B}} + \left[\vec{y}\right]_{\mathcal{B}} \qquad //$$

Ex Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$ given by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) if $\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, find $\left[\vec{x}\right]_{\mathcal{B}}$

(b) if $\left[\vec{y}\right]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ find $\vec{y}$

Solution 1

(a) By inspection $\vec{x} = \vec{v}_1 + \vec{v}_2$ so $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $y = 2\vec{v}_1 + \vec{v}_2 = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

Solution 2 (gives a taste of what's to come).

Preparation:

Let $S = \left[\vec{v}_1 \mid \vec{v}_2 \right] = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$.

Note $\det(S) = -5$ so $S^{-1} = \frac{-1}{5}\begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{bmatrix}$

For any a, b we have

$$S\begin{bmatrix} a \\ b \end{bmatrix} = \left[\vec{v}_1 \mid \vec{v}_2 \right]\begin{bmatrix} a \\ b \end{bmatrix}$$

$$= a\vec{v}_1 + b\vec{v}_2$$

If $\vec{x} = a\vec{v}_1 + b\vec{v}_2$ then $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}$ so

$$\boxed{S[\vec{x}]_{\mathcal{B}} = \vec{x}}$$ gives the vector in terms of the standard basis

$$\Rightarrow \boxed{[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}}$$

S and S^{-1} go between the standard basis and $\mathcal{B}$. In our example

sol'n to (b): $\vec{y} = S[\vec{y}]_{\mathcal{B}} = S\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix}$ ✓

sol'n to (a)

$$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x} = \begin{bmatrix}-1/5 & 3/5\\2/5 & -1/5\end{bmatrix}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} ✓$$

we'll also come back to this!